Linear systems theory
What is a system?

- Black-box description of characteristic behaviour

$$y(t) = F[x(t)]$$

known unknown known
Linear Systems Theory

Superposition principle (SPP)

- If input $x_1(t)$ \rightarrow output $y_1(t)$
- If input $x_2(t)$ \rightarrow output $y_2(t)$
- Then: $\alpha x_1(t) + \beta x_2(t)$ \rightarrow $\alpha y_1(t) + \beta y_2(t)$

In general:

$$\sum_{n=1}^{N} \alpha_n x_n(t) \rightarrow \sum_{n=1}^{N} \alpha_n y_n(t)$$

This is true for all $t$ !!!

$$y(t) = F[x(t)]$$
**Surprise 1: what looks linear might not be linear**

\[ y(t) = ax(t) + b = H_2(x(t)) \]

\[ y(t) = ax(t) = H_1(x(t)) \]

\[ H_1(x_1 + x_2) = ax_1 + ax_2 \]

\[ H_2(x_1 + x_2) = ax_1 + ax_2 + b \neq H(x_1) + H(x_2) = ax_1 + ax_2 + 2b \]
Central concept: Impulse response of a Linear System

Surprise 2: a linear system can completely change the shape of the input signal!!!

$\delta(t) = \begin{cases} 
\infty & \text{for } t = 0 \\
0 & \text{elsewhere}
\end{cases}$

such that $\int_{-\infty}^{\infty} \delta(t) \, dt = 1$ and $f(t_0) = \int_{-\infty}^{\infty} \delta(t-t_0) f(t) \, dt$

As a result of the SPP the response of the Linear System to an arbitrary input can be computed from the system’s impulse response!
Linear Systems Theory

Any signal can be decomposed into a series of pulses

\[ \delta(t) = \begin{cases} 
\infty & \text{for } t = 0 \\
0 & \text{elsewhere} 
\end{cases} \]

such that

\[ \int_{-\infty}^{\infty} \delta(t) dt = 1 \text{ and } f(t_0) = \int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt \]
Central concept: **Impulse response of a Linear System**

- How is this useful?
  - Precise description of signal $x(t)$ by Dirac impulses
    \[ x(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot \delta(t - \tau) \cdot d\tau \]
  - Precise description of response $y(t)$ from the impulse response
    \[ y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t - \tau) \cdot d\tau \]
  - Considering only causal systems, i.e. $h(t - \tau) = 0$ for $\tau \geq t$
    \[ y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t - \tau) \cdot d\tau \]
  - Change of variables:
    \[ y(t) = \int_{0}^{+\infty} x(t - \tau) \cdot h(\tau) \cdot d\tau \]
What does the convolution integral mean?

\[ y(t) = \int_{0}^{\infty} x(t - \tau) \cdot h(\tau) \cdot d\tau \]

- Current system’s output
- The start of the universe
- The past
- Superposition over the past
- System’s past input
- The system’s dynamic memory of the instantaneous past input
Exercise

Suppose you have $x(t) = \sin(\omega t)$ and assume impulse response $h(t)$. What is the system’s output?

- We need: $\sin(t - \tau) = \sin(t) \cos(\tau) - \cos(t) \sin(\tau)$

- Answer: $y(t) = G(\omega) \cdot \sin(\omega t + \varphi(\omega))$

Thus the output is a harmonic function again!
- Amplitude and phase depend on frequency of input
- But output frequency has not changed!
- Harmonic functions are Eigenfunctions of linear systems!!!
Alternative characterization of linear systems

- Amplitude characteristic $G(\omega)$
- Phase characteristic $\varphi(\omega)$

Together, they provide the transfer characteristic!

- $H(\omega)$
- Fourier analysis: $H(\omega)$ is the Fourier transform of $h(\tau)$!

$$Y(\omega) = H(\omega) \cdot X(\omega)$$

$$y(t) = G(\omega) \cdot \sin(\omega t + \varphi(\omega))$$
Problem: the Fourier transform of many often used functions is not defined!

E.g. step function

\[ U(\omega) = \int_{-\infty}^{\infty} U(t)e^{-i\omega t} dt = \int_{0}^{\infty} e^{-i\omega t} dt = -\frac{1}{i\omega}e^{-i\omega t}\bigg|_{0}^{\infty} \]
Solution: Laplace transform!

- Integral transform
- Resolves a function or signal into its moments
  - e.g. statistical moments (mean, variance, etc.)
  - Modes of vibration (frequencies)
- Transform between time and frequency
- Compact systems description

Definition:

\[ F(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} \cdot f(t) dt \]

\( s: \) complex number
\( s = \sigma + i\omega \)
Linear Systems Theory

- Laplace transform
We can now solve our problem...

- Step function has a solution in Laplace space!

\[ \mathcal{L}[U(t)] = \int_0^\infty U(t)e^{-st}dt = -\frac{1}{s}e^{-st}\bigg|_0^\infty = \frac{1}{s} \]
### Laplace transform

<table>
<thead>
<tr>
<th></th>
<th>Time domain</th>
<th>'s' domain</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linearity</strong></td>
<td>$af(t) + bg(t)$</td>
<td>$aF(s) + bG(s)$</td>
<td>Can be proved using basic rules of integration.</td>
</tr>
<tr>
<td><strong>Frequency</strong></td>
<td>$tf(t)$</td>
<td>$-F'(s)$</td>
<td>$F'$ is the first derivative of $F$.</td>
</tr>
<tr>
<td><strong>differentiation</strong></td>
<td>$t^n f(t)$</td>
<td>$(-1)^n F^{(n)}(s)$</td>
<td>More general form, $n$th derivative of $F(s)$.</td>
</tr>
<tr>
<td><strong>Differentiation</strong></td>
<td>$f'(t)$</td>
<td>$sF(s) - f(0)$</td>
<td>$f$ is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts.</td>
</tr>
<tr>
<td><strong>Second</strong></td>
<td>$f''(t)$</td>
<td>$s^2 F(s) - sf(0) - f'(0)$</td>
<td>$f$ is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.</td>
</tr>
<tr>
<td><strong>General</strong></td>
<td>$f^{(n)}(t)$</td>
<td>$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$</td>
<td>$f$ is assumed to be $n$-times differentiable, with $n$th derivative of exponential type. Follows by mathematical induction.</td>
</tr>
<tr>
<td><strong>Frequency</strong></td>
<td>$\frac{f(t)}{t}$</td>
<td>$\int_s^\infty F(\sigma) , d\sigma$</td>
<td></td>
</tr>
<tr>
<td><strong>Integration</strong></td>
<td>$\int_0^t f(\tau) , d\tau = (u * f)(t)$</td>
<td>$\frac{1}{s} F(s)$</td>
<td>$u(t)$ is the Heaviside step function. Note $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.</td>
</tr>
<tr>
<td><strong>Scaling</strong></td>
<td>$f(at)$</td>
<td>$\frac{1}{</td>
<td>a</td>
</tr>
<tr>
<td><strong>Frequency</strong></td>
<td>$e^{-at} f(t)$</td>
<td>$F(s - a)$</td>
<td>$u(t)$ is the Heaviside step function</td>
</tr>
<tr>
<td><strong>Time shifting</strong></td>
<td>$f(t-a) u(t-a)$</td>
<td>$e^{-as} F(s)$</td>
<td>$u(t)$ is the Heaviside step function</td>
</tr>
<tr>
<td><strong>Convolution</strong></td>
<td>$(f * g)(t)$</td>
<td>$F(s) \cdot G(s)$</td>
<td>$f(t)$ and $g(t)$ are extended by zero for $t &lt; 0$ in the definition of the convolution.</td>
</tr>
<tr>
<td>** Periodic Function**</td>
<td>$f(t)$</td>
<td>$\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) , dt$</td>
<td>$f(t)$ is a periodic function of period $T$ so that $f(t) = f(t + T)$, $\forall t \geq 0$. This is the result of the time shifting property and the geometric series.</td>
</tr>
</tbody>
</table>
## Laplace transform

<table>
<thead>
<tr>
<th>ID</th>
<th>Function</th>
<th>Time domain ( f(t) = \mathcal{L}^{-1} { F(s) } )</th>
<th>Laplace s-domain ( F(s) = \mathcal{L} { f(t) } )</th>
<th>Region of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>ideal delay ( \delta(t - \tau) )</td>
<td>( e^{-\tau s} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1a</td>
<td>unit impulse ( \delta(t) )</td>
<td>( 1 )</td>
<td>all ( s )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>delayed nth power with frequency shift ( (t - \tau)^n e^{-\alpha(t-\tau)} \cdot u(t - \tau) )</td>
<td>( \frac{e^{-\tau s}}{(s + \alpha)^{n+1}} )</td>
<td>( \text{Re}{s} &gt; -\alpha )</td>
<td></td>
</tr>
<tr>
<td>2a</td>
<td>nth power (for integer ( n )) ( \frac{t^n}{n!} \cdot u(t) )</td>
<td>( \frac{1}{s^{n+1}} )</td>
<td>( \begin{align*} \text{Re}{s} &amp; &gt; 0 \ n &amp; &gt; -1 \end{align*} )</td>
<td></td>
</tr>
<tr>
<td>2a.1</td>
<td>nth power (for complex ( \alpha )) ( \frac{t^q}{\Gamma(q + 1)} \cdot u(t) )</td>
<td>( \frac{1}{s^{q+1}} )</td>
<td>( \begin{align*} \text{Re}{s} &amp; &gt; 0 \ \text{Re}{q} &amp; &gt; -1 \end{align*} )</td>
<td></td>
</tr>
<tr>
<td>2a.2</td>
<td>unit step ( u(t) )</td>
<td>( \frac{1}{s} )</td>
<td>( \text{Re}{s} &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>2b</td>
<td>delayed unit step ( u(t - \tau) )</td>
<td>( e^{-\tau s} )</td>
<td>( \text{Re}{s} &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>2c</td>
<td>ramp ( t \cdot u(t) )</td>
<td>( \frac{1}{s^2} )</td>
<td>( \text{Re}{s} &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>2d</td>
<td>nth power with frequency shift ( \frac{t^n e^{-\alpha t}}{n!} \cdot u(t) )</td>
<td>( \frac{1}{(s + \alpha)^{n+1}} )</td>
<td>( \text{Re}{s} &gt; -\alpha )</td>
<td></td>
</tr>
<tr>
<td>2d.1</td>
<td>exponential decay ( e^{-\alpha t} \cdot u(t) )</td>
<td>( \frac{1}{s + \alpha} )</td>
<td>( \text{Re}{s} &gt; -\alpha )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>exponential approach ( (1 - e^{-\alpha t}) \cdot u(t) )</td>
<td>( \frac{\alpha}{s(s + \alpha)} )</td>
<td>( \text{Re}{s} &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>sine ( \sin(\omega t) \cdot u(t) )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
<td>( \text{Re}{s} &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>cosine ( \cos(\omega t) \cdot u(t) )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
<td>( \text{Re}{s} &gt; 0 )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>hyperbolic sine ( \sinh(\alpha t) \cdot u(t) )</td>
<td>( \frac{\alpha}{s^2 - \alpha^2} )</td>
<td>( \text{Re}{s} &gt; \vert \alpha \vert )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>hyperbolic cosine ( \cosh(\alpha t) \cdot u(t) )</td>
<td>( \frac{s}{s^2 - \alpha^2} )</td>
<td>( \text{Re}{s} &gt; \vert \alpha \vert )</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Exponentially-decaying sine wave ( e^{-\alpha t} \sin(\omega t) \cdot u(t) )</td>
<td>( \frac{\omega}{(s + \alpha)^2 + \omega^2} )</td>
<td>( \text{Re}{s} &gt; \alpha )</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Exponentially-decaying cosine wave ( e^{-\alpha t} \cos(\omega t) \cdot u(t) )</td>
<td>( \frac{s + \alpha}{(s + \alpha)^2 + \omega^2} )</td>
<td>( \text{Re}{s} &gt; \alpha )</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>nth root ( \sqrt[n]{t} \cdot u(t) )</td>
<td>( s^{-(n+1)/n} \cdot \Gamma \left( 1 + \frac{1}{n} \right) )</td>
<td>( \text{Re}{s} &gt; 0 )</td>
<td></td>
</tr>
</tbody>
</table>
State space representations

- Laplace transform

  - Example: solving

    \[
    \frac{dn(t)}{dt} = -\lambda n(t)
    \]

    - Laplace transform:

      \[
      (sN(s) - n(0)) + \lambda N(s) = 0
      \]

      \[
      \iff N(s) = \frac{n(0)}{s + \lambda}
      \]

    - Inverse Laplace transform:

      \[
      n(t) = \mathcal{L}^{-1}(N(s)) = n(0) \cdot e^{-\lambda t}
      \]
The role of feedback
Feedback

- If we have a system $G$ controlling a plant $P$

![Diagram](image)

- Cascade of 2 linear systems
- Transfer function $\frac{Y(s)}{X(s)} = G(s) \cdot P(s)$

Example

- $P$ is an elastic band (exponential decay)
- $P(s) = \frac{1}{Ts+1}$ with time constant $T$
- If $G(s) = \text{constant} = K$, then
  $\frac{Y(s)}{X(s)} = \frac{K}{Ts + 1}$
Feedback

Now let’s add (negative) feedback…!

\[
\frac{Y(s)}{X(s)} = \frac{G(s) \cdot P(s)}{1 + H(s) \cdot G(s) \cdot P(s)}
\]

Example

If \( H(s) = 1 \), then

\[
\frac{Y(s)}{X(s)} = \frac{G_{fb}}{T_{fb} s + 1}
\]

With \( G_{fb} = \frac{K}{1+K} \) and \( T_{fb} = \frac{T}{1+K} \)

Result: smaller time constant = faster response!!!
Feedback

- Now let’s add (negative) feedback…!

\[
Y(s) = \frac{G(s) \cdot P(s)}{1 + H(s) \cdot G(s) \cdot P(s)}
\]

- Example

- If \( H(s) \cdot G(s) \cdot P(s) \gg 1 \), then \[
\frac{Y(s)}{X(s)} = \frac{1}{H(s)}
\]

- Result: **system is independent of feed-forward path!!!**
- I.e. feedback ensures that the total system is still highly reliable! (even for vulnerable systems, e.g. fatigue, large gain changes…)}
Feedback

- Now let’s add (negative) feedback…!

\[
\frac{Y(s)}{X(s)} = \frac{G(s) \cdot P(s)}{1 + H(s) \cdot G(s) \cdot P(s)}
\]

- Transfer function

- Example

<table>
<thead>
<tr>
<th>H</th>
<th>total system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrator</td>
<td>Differentiator</td>
</tr>
<tr>
<td>Differentiator</td>
<td>Integrator</td>
</tr>
<tr>
<td>Low-pass filter</td>
<td>High pass filter</td>
</tr>
<tr>
<td>High-pass filter</td>
<td>Low pass filter</td>
</tr>
</tbody>
</table>
Feedback

External perturbations?

Without feedback: \( \frac{Y(s)}{W(s)} = P(s) = \frac{1}{Ts+1} \)

With feedback: \( \frac{Y(s)}{W(s)} = \frac{P(s)}{1+KP(s)} = \frac{G_{pert}}{T_{fb}s+1} \)

With \( G_{pert} = \frac{1}{1+K} \)

Result: gain to external perturbation is reduced by \( \approx \frac{1}{K} \) for \( K \gg 1 \)
Feedback

Other advantages of feedback:
- Reduced sensitivity of a system to parameter variations
- Potential to stabilize unstable systems…
Stability, zeros & poles
Definitions

- **Poles**: values of complex variable $s$ for which the transfer function becomes infinite
- **Zeros**: values of complex variable $s$ for which the transfer function becomes zero

Example: \[ G(s) = \frac{10(s+2)}{s(s+1)(s+3)} \]

- $G(s)$ has once zero at $s=-2$ and three poles at $s=0$, $s=-1$ and $s=-3$
Definitions

- **Stability**: A system is stable if the output is bounded for any bounded input.

- **Criterion for stability**: The real portion of all poles must be negative!

Example: \( G(s) = \frac{P(s)}{(s + a_1)(s + a_2)\ldots(s + a_n)} = \frac{K_1}{s + a_1} + \frac{K_2}{s + a_2} + \ldots + \frac{K_n}{s + a_n} \)

This corresponds to \( \mathcal{L}^{-1}[G(s)] = K_1 e^{-a_1t} + K_2 e^{-a_2t} + \ldots + K_n e^{-a_nt} \)

- For \( a_i > 0 \), poles are negative, reflecting **decaying** exponentials = STABLE
- For \( a_i < 0 \), poles are positive, reflecting **rising** exponentials = UNSTABLE
More on linear systems
State space representations

- Linear systems theory
  \[
  \dot{x}(t) = A(t)x(t) + B(t)u(t) \\
y(t) = C(t)x(t) + D(t)u(t)
  \]

- System with \( p \) inputs, \( q \) outputs and \( n \) state variables
  - \( x \): state vector \( \in \mathbb{R}^n \)
  - \( y \): output vector \( \in \mathbb{R}^q \)
  - \( u \): input (control) vector \( \in \mathbb{R}^p \)
  - \( A \): state matrix \( (n \times n) \)
  - \( B \): input matrix \( (n \times p) \)
  - \( C \): output matrix \( (q \times n) \)
  - \( D \): feed-through matrix \( (q \times p) \)

- In continuous time-invariant models, all matrices are constant
State space representations

- **Transfer functions**
  - Back to our linear system…
    \[
    \dot{x}(t) = A(t)x(t) + B(t)u(t) \]
    \[
    y(t) = C(t)x(t) + D(t)u(t) \]
  - Laplace transform of \( x \) yields:
    \[
    sX(s) = AX(s) + BU(s) \]
  - Solving for \( X \):
    \[
    X(s) = \frac{BU(s)}{sI - A} \]
  - Similarly for \( Y \):
    \[
    Y(s) = C\frac{BU(s)}{sI - A} + DU(s) \]

- **Transfer function \( G \):**
  \[
  G(s) \equiv \frac{Y(s)}{U(s)} = C\frac{B}{sI - A} + D \]
  (ratio of output to input of system)
State space representations

- **Transfer functions**
  - Transfer function $G$:
    \[ G(s) \equiv \frac{Y(s)}{U(s)} = C \frac{B}{sI - A} + D \]
    - $G$ is $(q \times p)$ matrix
    - For every input, there are $q$ transfer functions, i.e. one for each output

- Simple representation of input-output mapping

- **Examples**
  - If $B, C = I$ and $A, D = 0$, then $y$ = integral of $u$
  - If $A = I$, then exponential

![Diagram](Wikipedia)
State space representations

- **Linear systems theory**
  \[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]
  \[ y(t) = C(t)x(t) + D(t)u(t) \]

- **Link to neural networks (linear time-invariant models)**
Linear systems theory
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) \\
y(t) = C(t)x(t) + D(t)u(t)
\]

Controllability
- It is possible (by admissible inputs) to steer the states from any initial value to any final value within some finite time window.
- Continuous time-invariant models are controllable if

\[
\text{rank} \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = n
\]

(rank = number of linearly independent rows in the matrix)

# state variables
State space representations

- Linear systems theory
  \[
  \dot{x}(t) = A(t)x(t) + B(t)u(t) \\
  y(t) = C(t)x(t) + D(t)u(t)
  \]

- Observability
  - A measure of how well internal states of a system can be inferred by knowledge of its external outputs.
  - Continuous time-invariant models are observable if
  \[
  \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n
  \]
Next up: Saccades