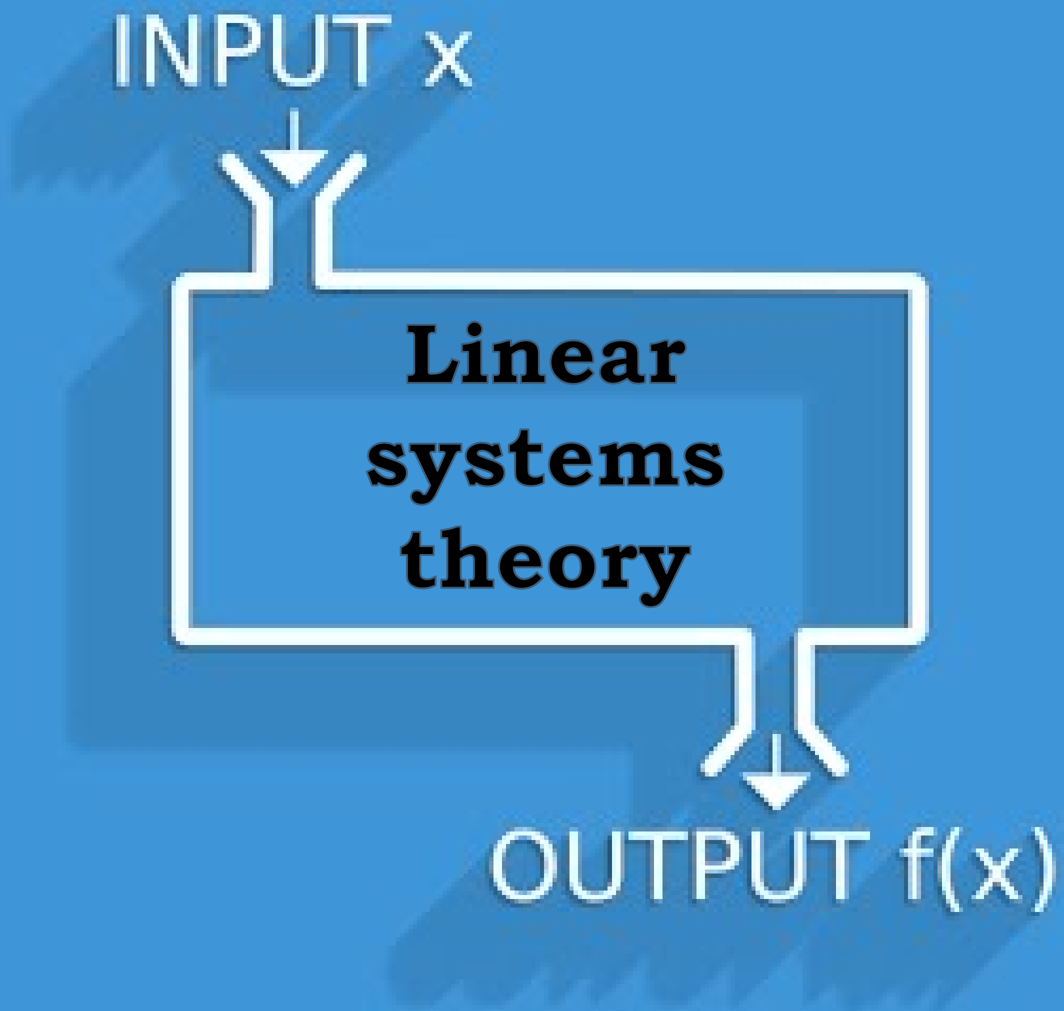
Linear systems theory

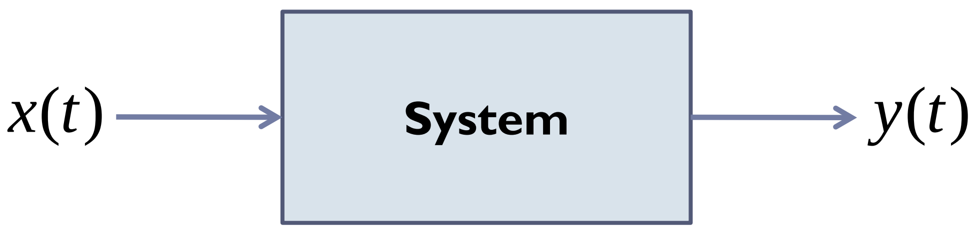


CoSMo 2017
Gunnar Blohm

Linear Systems Theory

What is a system?

- Black-box description of characteristic behaviour



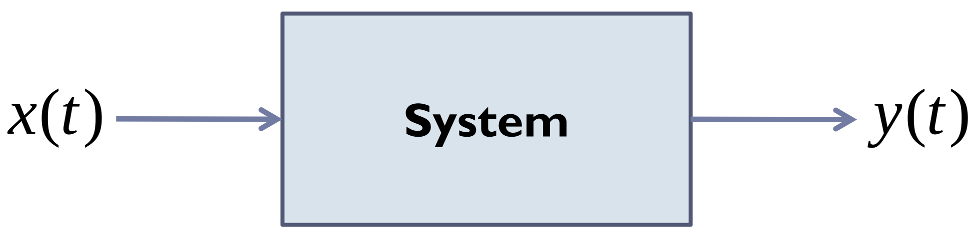
$$y(t) = F[x(t)]$$

known **unknown** **known**

Linear Systems Theory

Superposition principle (SPP)

- If input $x_1(t)$ → output $y_1(t)$
- If input $x_2(t)$ → output $y_2(t)$
- Then: $\alpha\, x_1(t) + \beta\, x_2(t)$ → $\alpha\, y_1(t) + \beta\, y_2(t)$



$$y(t) = F[x(t)]$$

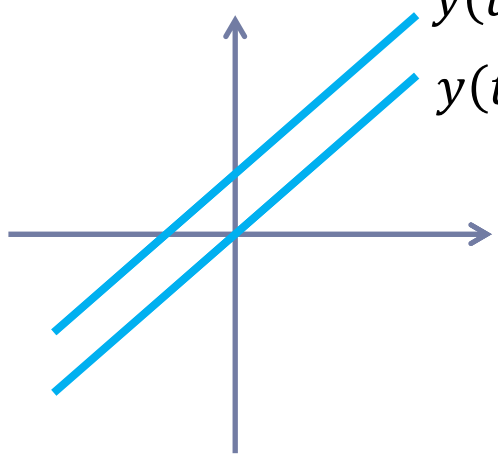
- **In general:** $$\sum_{n=1}^{N} \alpha_n x_n(t) \rightarrow \sum_{n=1}^{N} \alpha_n y_n(t)$$
  - This is true for all t !!!

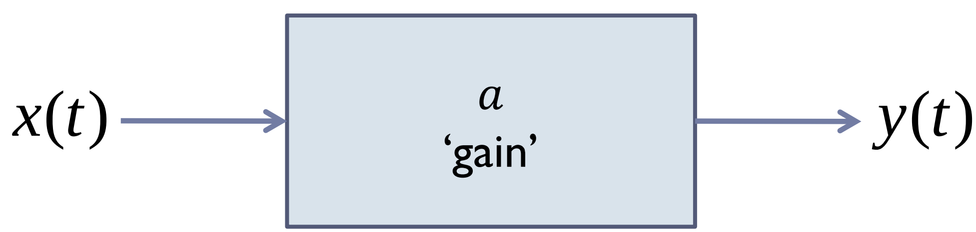
Linear Systems Theory

- Surprise 1: what looks linear might not be linear

$$y(t) = ax(t) + b = H_2(x(t))$$

$$y(t) = ax(t) = H_1(x(t))$$





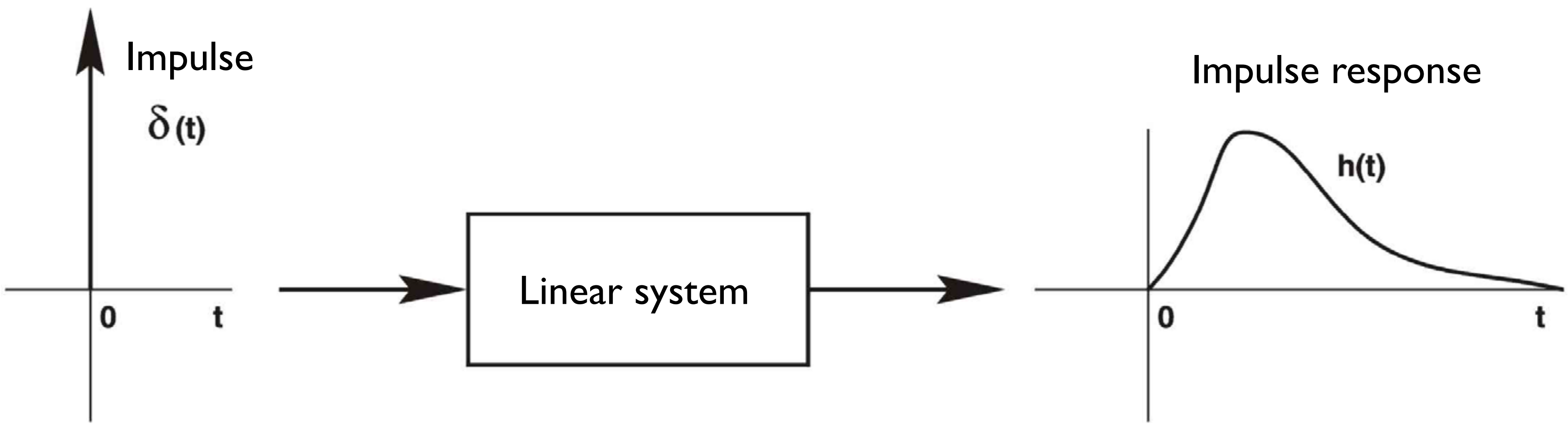
$$H_1(x_1 + x_2) = ax_1 + ax_2$$

$$H_2(x_1 + x_2) = ax_1 + ax_2 + \mathrm{b} \neq H(x_1) + H(x_2) = ax_1 + ax_2 + 2\mathrm{b}$$

Linear Systems Theory

Central concept: Impulse response of a Linear System

Surprise 2: a linear system can completely change the shape of the input signal!!!

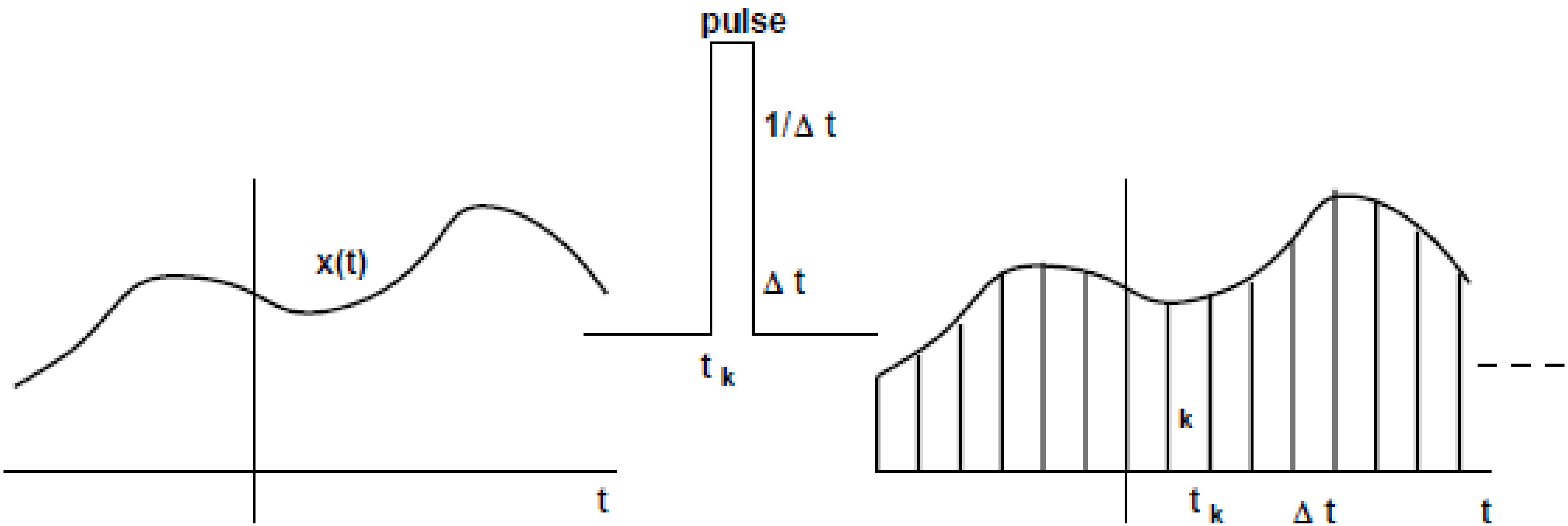


$$\delta(t)=\begin{cases}\infty \text{ for } t=0\\ 0 \text{ elsewhere}\end{cases} \quad \text{such that} \int_{-\infty}^{\infty}\delta(t)\,dt=1 \text{ and } f(t_0)=\int_{-\infty}^{\infty}\delta(t-t_0)f(t)\,dt$$

As a result of the SPP the response of the Linear System to an arbitrary input can be computed from the system's impulse response!

Linear Systems Theory

Any signal can be decomposed into a series of pulses



$$\delta(t) = \begin{cases} \infty \text{ for } t = 0 \\ 0 \text{ elsewhere} \end{cases} \quad \text{such that} \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \text{ and } f(t_0) = \int_{-\infty}^{\infty} \delta(t - t_0) f(t)\,dt$$

Linear Systems Theory

Central concept: Impulse response of a Linear System

- How is this useful?
  - Precise description of signal x(t) by Dirac impulses

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot \delta(t-\tau) \cdot d\tau$$

  - Precise description of response y(t) from the impulse response

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

  - Considering only causal systems, i.e. $h(t-\tau) = 0 \;\; for \; \tau \geq t$

$$y(t) = \int_{-\infty}^{t} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

  - Change of variables: $$y(t) = \int_{0}^{+\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

Convolution Integral

Linear Systems Theory

What does the convolution integral mean?

Current system's output

The start of the universe

The past

Superposition over the past

$$y(t) = \int_{0}^{+\infty} x(t-\tau) \cdot h(\tau) \cdot d\tau$$

now

now

System's past input

The system's dynamic memory of the instantaneous past input

Linear Systems Theory

- Exercise
  - Suppose you have x(t) = sin(ωt) and assume impulse response h(t). What is the system's output?
    - We need: $\sin(t-\tau) = \sin(t)\cos(\tau) - \cos(t)\sin(\tau)$
  - Answer: $y(t) = G(\omega) \cdot \sin(\omega t + \varphi(\omega))$
  - Thus the output is a harmonic function again!
    - Amplitude and phase depend on frequency of input
    - But output frequency has not changed!
    - Harmonic functions are Eigenfunctions of linear systems!!!

Linear Systems Theory

- Alternative characterization of linear systems
  - Amplitude characteristic $G(\omega)$
  - Phase characteristic $\varphi(\omega)$

$$y(t) = G(\omega) \cdot \sin(\omega t + \varphi(\omega))$$

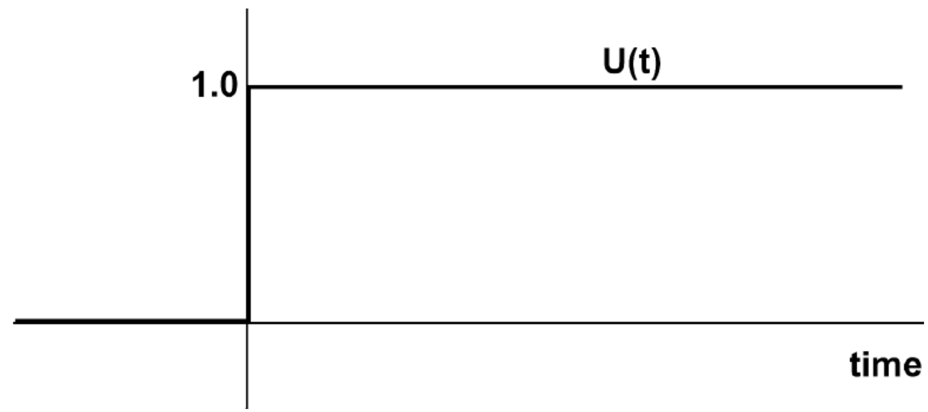
- Together, they provide the **transfer characteristic**!
  - H(ω)
  - Fourier analysis: H(ω) is the Fourier transform of h(τ)!

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

$$\boxed{Y(\omega) = H(\omega) \cdot X(\omega)}$$

Linear Systems Theory

- Problem: the Fourier transform of many often used functions is not defined!
  - E.g. step function



$$U(\omega) = \int_{-\infty}^{\infty} U(t) e^{-i\omega t} dt = \int_{0}^{\infty} e^{-i\omega t} dt = -\frac{1}{i\omega} e^{-i\omega t} \Big|_0^{\infty}$$

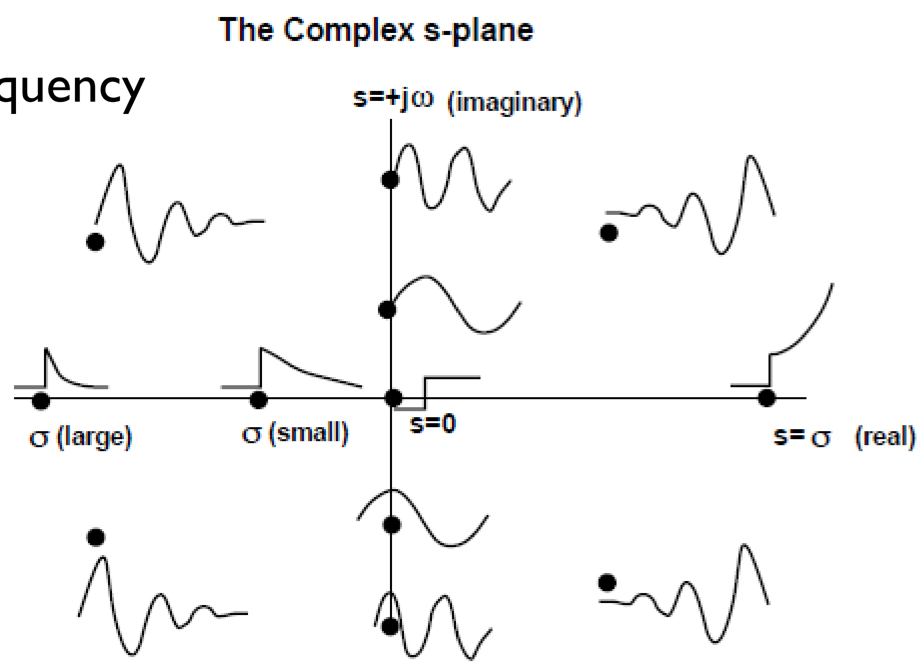
Linear Systems Theory

- Solution: **Laplace transform!**
  - Integral transform
  - Resolves a function or signal into its moments
    - e.g. statistical moments (mean, variance, etc…)
    - Modes of vibration (frequencies)
    - Transform between time and frequency
  - Compact systems description
  - Definition:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

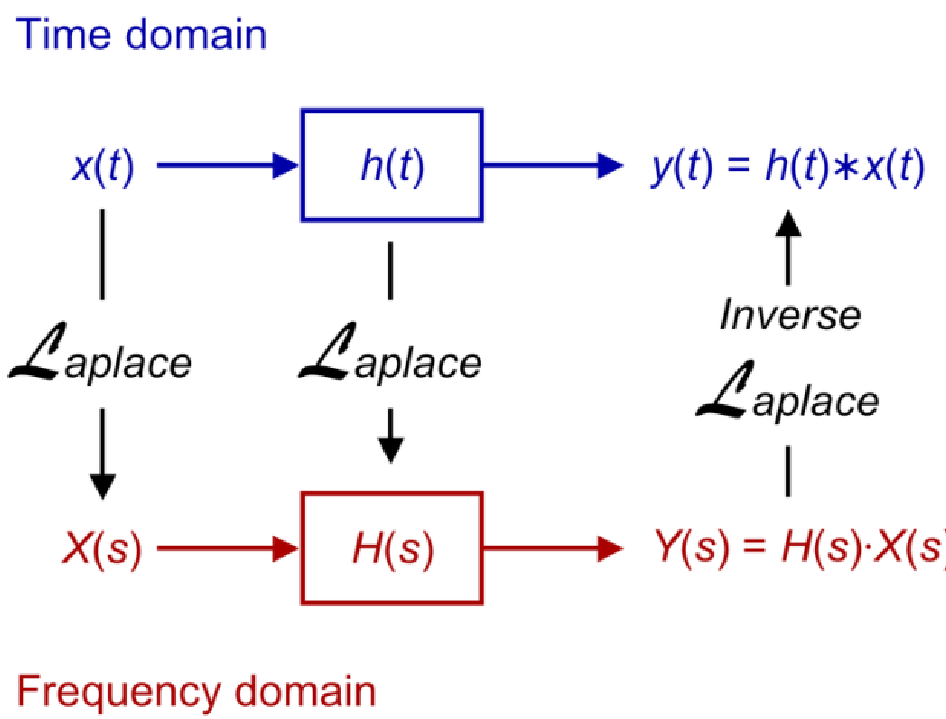
s: complex number

$$s = \sigma + i\omega$$



Linear Systems Theory

- Laplace transform



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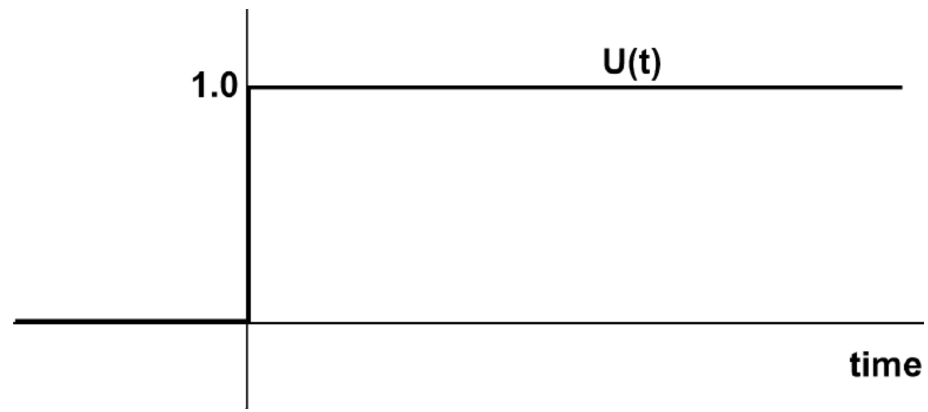
Linear Systems Theory

- We can now solve our problem…
  - Step function has a solution in Laplace space!



$$\mathcal{L}[U(t)] = \int_0^\infty U(t)e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

Linear Systems Theory

- Laplace transform

	Time domain	's' domain	Comment
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$	Can be proved using basic rules of integration.
Frequency differentiation	$tf(t)$	$-F'(s)$	F' is the first derivative of F.
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, (n)th derivative of F(s).
Differentiation	$f'(t)$	$sF(s)-f(0)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second Differentiation	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	f is assumed to be n-times differentiable, with n^{th} derivative of exponential type. Follow by mathematical induction.
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	
Integration	$\int_0^t f(\tau)\,d\tau=(u*f)(t)$	$\frac{1}{s}F(s)$	$u(t)$ is the Heaviside step function. Note $(u*f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	
Frequency shifting	$e^{at}f(t)$	$F(s-a)$	
Time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$	$u(t)$ is the Heaviside step function
Convolution	$(f*g)(t)$	$F(s)\cdot G(s)$	$f(t)$ and $g(t)$ are extended by zero for $t<0$ in the definition of the convolution.
Periodic Function	$f(t)$	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$	$f(t)$ is a periodic function of period T so that $f(t)=f(t+T),\ \forall t\geq 0$. This is the result of the time shifting property and the geometric series.

Wikipedia

Linear Systems Theory

- Laplace transform

ID	Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence
1	ideal delay	$\delta(t-\tau)$	$e^{-\tau s}$	
1a	unit impulse	$\delta(t)$	1	all s
2	delayed *n*th power with frequency shift	$\frac{(t-\tau)^n}{n!} e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{e^{-\tau s}}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$
2a	*n*th power (for integer *n*)	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$ $(n > -1)$
2a.1	*q*th power (for complex *q*)	$\frac{t^q}{\Gamma(q+1)} \cdot u(t)$	$\frac{1}{s^{q+1}}$	$\mathrm{Re}\{s\} > 0$ $(\mathrm{Re}\{q\} > -1)$
2a.2	unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
2b	delayed unit step	$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$\mathrm{Re}\{s\} > 0$
2c	ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
2d	*n*th power with frequency shift	$\frac{t^n}{n!} e^{-\alpha t} \cdot u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$
2d.1	exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}\{s\} > -\alpha$
3	exponential approach	$(1-e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\mathrm{Re}\{s\} > 0$
4	sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}\{s\} > 0$
5	cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}\{s\} > 0$
6	hyperbolic sine	$\sinh(\alpha t) \cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\mathrm{Re}\{s\} > \lvert\alpha\rvert$
7	hyperbolic cosine	$\cosh(\alpha t) \cdot u(t)$	$\frac{s}{s^2-\alpha^2}$	$\mathrm{Re}\{s\} > \lvert\alpha\rvert$
8	Exponentially-decaying sine wave	$e^{-\alpha t}\sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}\{s\} > \alpha$
9	Exponentially-decaying cosine wave	$e^{-\alpha t}\cos(\omega t) \cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}\{s\} > \alpha$
10	*n*th root	$\sqrt[n]{t} \cdot u(t)$	$s^{-(n+1)/n} \cdot \Gamma\left(1+\frac{1}{n}\right)$	$\mathrm{Re}\{s\} > 0$

Wikipedia

State space representations

- Laplace transform
  - Example: solving $\frac{dn(t)}{dt} = -\lambda n(t)$
    - Laplace transform: $\left(sN(s) - n(0)\right) + \lambda N(s) = 0$

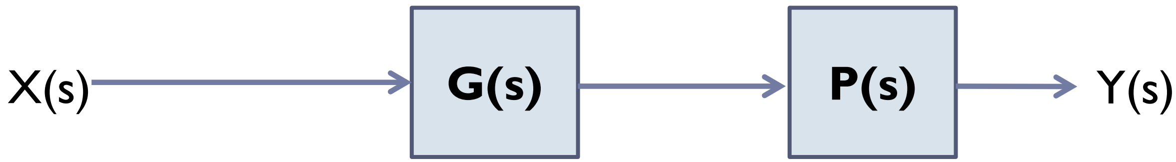
$$\Leftrightarrow \quad N(s) = \frac{n(0)}{s + \lambda}$$

    - Inverse Laplace transform: $n(t) = \mathcal{L}^{-1}\left(N(s)\right) = n(0) \cdot e^{-\lambda t}$

The role of feedback

Feedback

- If we have a system G controlling a plant P

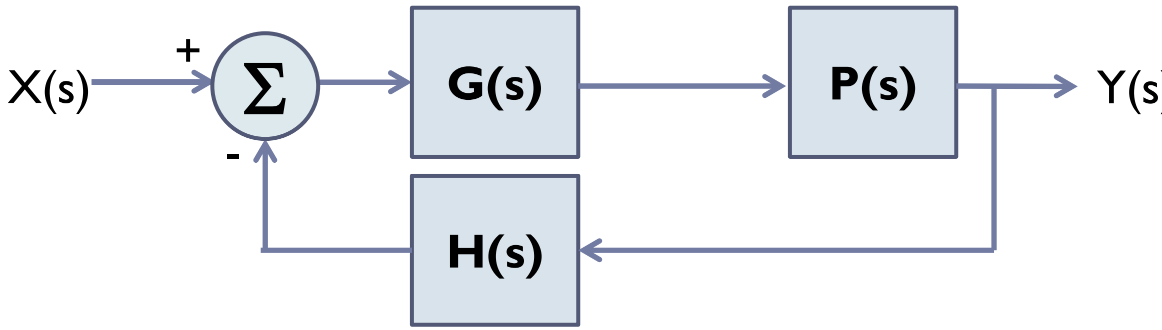


  - Cascade of 2 linear systems
  - Transfer function $\frac{Y(s)}{X(s)} = G(s) \cdot P(s)$
- Example
  - P is an elastic band (exponential decay)
  - $P(s) = \frac{1}{Ts+1}$ with time constant T
  - If G(s) = constant = K, then

$$\frac{Y(s)}{X(s)} = \frac{K}{Ts+1}$$

Feedback

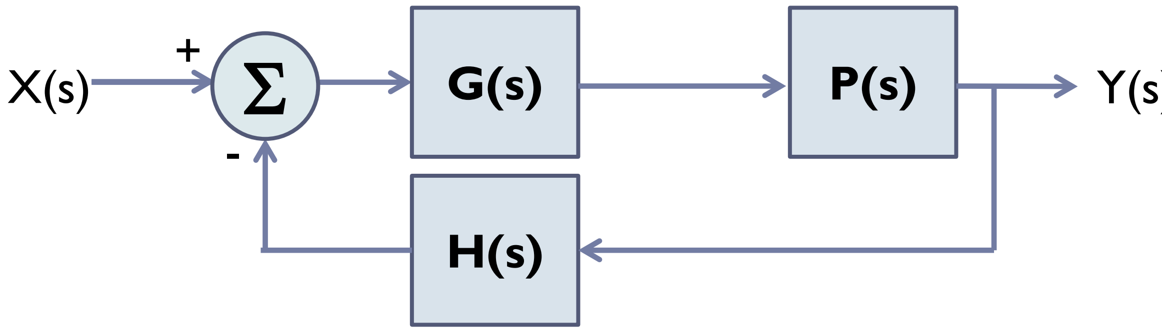
- Now let's add (negative) feedback…!



  - Transfer function $\frac{Y(s)}{X(s)} = \frac{G(s) \cdot P(s)}{1 + H(s) \cdot G(s) \cdot P(s)}$
- Example
  - If H(s) = 1, then $\frac{Y(s)}{X(s)} = \frac{G_{fb}}{T_{fb}s+1}$
  - With $G_{fb} = {}^{K}/_{(1+K)}$ and $T_{fb} = {}^{T}/_{(1+K)}$
  - Result: **smaller time constant = faster response!!!**

Feedback

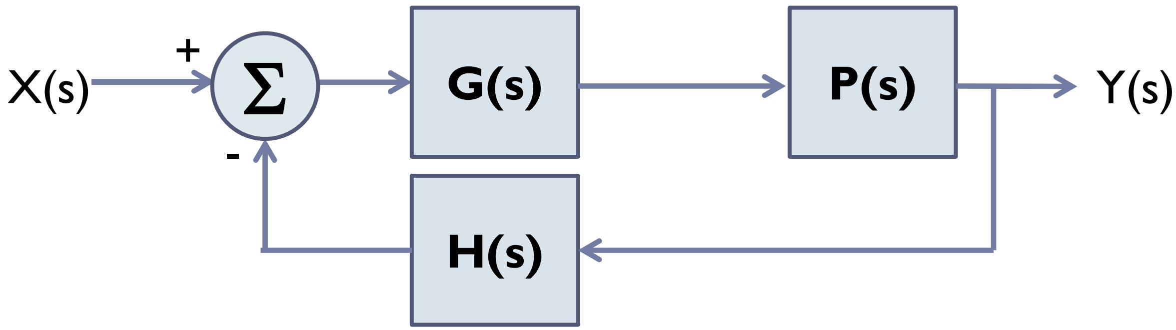
- Now let's add (negative) feedback…!



  - Transfer function $\frac{Y(s)}{X(s)} = \frac{G(s) \cdot P(s)}{1 + H(s) \cdot G(s) \cdot P(s)}$
- Example
  - If $H(s) \cdot G(s) \cdot P(s) \gg 1$, then $\frac{Y(s)}{X(s)} = \frac{1}{H(s)}$!!!
  - Result: **system is independent of feed-forward path!!!**
  - I.e. feedback ensures that the total system is still highly reliable! (even for vulnerable systems, e.g. fatigue, large gain changes…)

Feedback

- Now let's add (negative) feedback…!

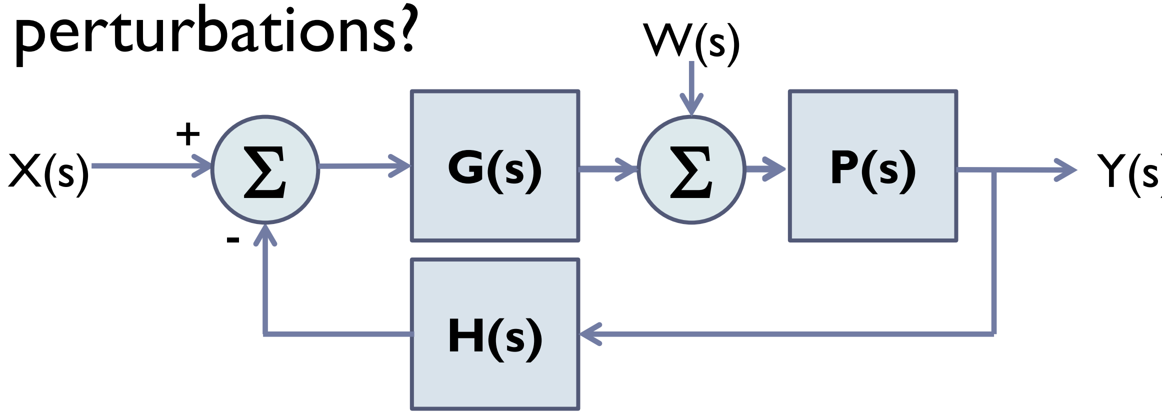


  - Transfer function $\frac{Y(s)}{X(s)} = \frac{G(s) \cdot P(s)}{1 + H(s) \cdot G(s) \cdot P(s)}$
- Example
  - H total system

H		total system
Integrator	⇔	Differentiator
Differentiator	⇔	Integrator
Low-pass filter	⇔	High pass filter
High-pass filter	⇔	Low pass filter

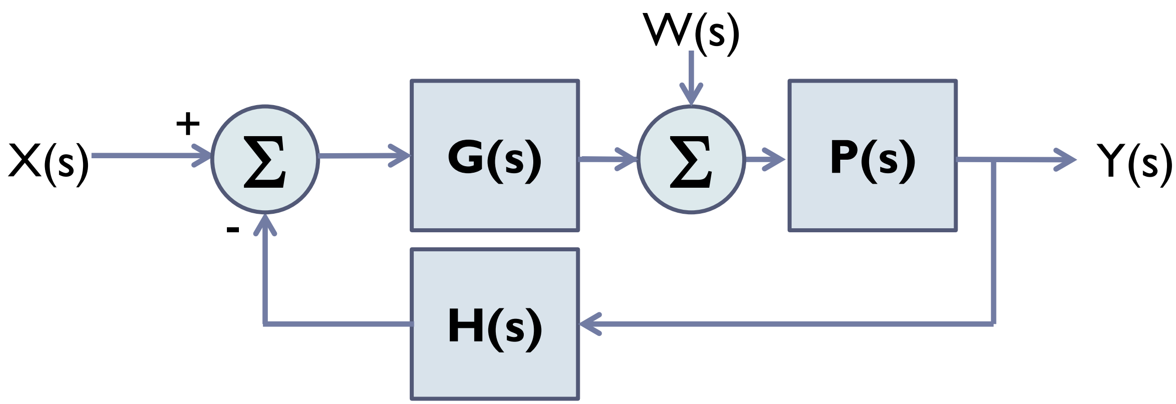
Feedback

- External perturbations?



- Without feedback: $\frac{Y(s)}{W(s)} = \mathrm{P(s)} = \frac{1}{Ts+1}$
- With feedback: $\frac{Y(s)}{W(s)} = \frac{P(s)}{1+KP(s)} = \frac{G_{pert}}{T_{fb}s+1}$
- With $G_{pert} = {}^{1}/_{(1+K)}$
- Result: **gain to external perturbation is reduced by** $\approx {}^{1}/_{K}$ **for K>>1**

Feedback



- Other advantages of feedback:
  - Reduced sensitivity of a system to parameter variations
  - Potential to stabilize unstable systems…

Stability, zeros & poles

Definitions

- **Poles**: values of complex variable s for which the transfer function becomes infinite
- **Zeros**: values of complex variable s for which the transfer function becomes zero

- Example: $$G(s) = \frac{10(s+2)}{s(s+1)(s+3)}$$

  - G(s) has once zero at s=-2 and three poles at s=0, s=-1 and s=-3

Definitions

- **Stability**: A system is stable if the output is bounded for any bounded input
  - **Criterion for stability**: The real portion of all poles must be negative!
  - Example: $G(s) = \frac{P(s)}{(s+a_1)(s+a_2)....(s+a_n)} = \frac{K_1}{(s+a_1)} + \frac{K_2}{(s+a_2)} + ... \frac{K_n}{(s+a_n)}$
    - This corresponds to $\mathcal{L}^{-1}\left[G(s)\right] = K_1 e^{-a_1 t} + K_2 e^{-a_2 t} + ... K_n e^{-a_n t}$
      - For ai>0, poles are negative, reflecting **decaying** exponentials = STABLE
      - For ai<0, poles are positive, reflecting **rising** exponentials = UNSTABLE

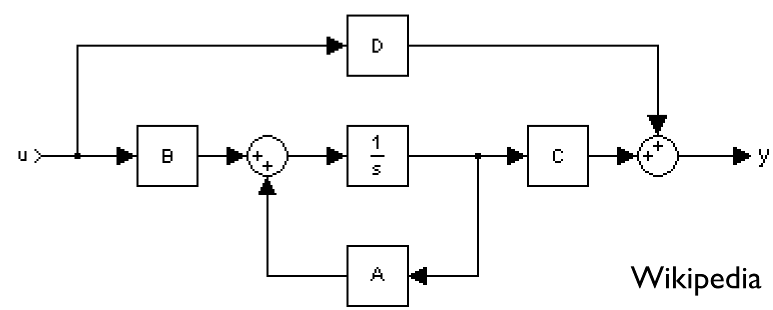
More on linear systems

State space representations

- Linear systems theory

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

  - System with p inputs, q outputs and n state variables
    - **x**: state vector $\in \Re^n$
    - **y**: output vector $\in \Re^q$
    - **u**: input (control) vector $\in \Re^p$
    - **A**: state matrix (n x n)
    - **B**: input matrix (n x p)
    - **C**: output matrix (q x n)
    - **D**: feed-through matrix (q x p)



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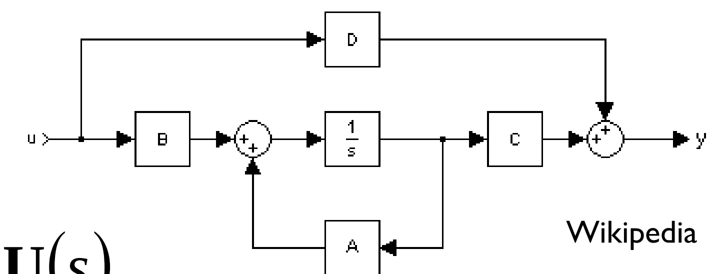
  - In continuous time-invariant models, all matrices are constant

State space representations

- Transfer functions
  - Back to our linear system… $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$
    $\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$
  - Laplace transform of x yields: $s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$
  - Solving for **X**: $\mathbf{X}(s) = \dfrac{\mathbf{B}\mathbf{U}(s)}{s\mathbf{I} - \mathbf{A}}$
  - Similarly for **Y**: $\mathbf{Y}(s) = \mathbf{C}\dfrac{\mathbf{B}\mathbf{U}(s)}{s\mathbf{I} - \mathbf{A}} + \mathbf{D}\mathbf{U}(s)$



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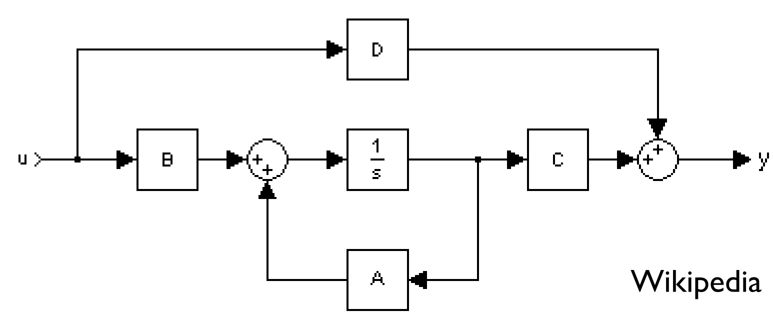
  - Transfer function **G**: $\mathbf{G}(s) \equiv \dfrac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}\dfrac{\mathbf{B}}{s\mathbf{I} - \mathbf{A}} + \mathbf{D}$

    (ratio of output to input of system)

State space representations

- Transfer functions
  - Transfer function **G**: $$\mathbf{G}(s) \equiv \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}\frac{\mathbf{B}}{s\mathbf{I}-\mathbf{A}} + \mathbf{D}$$
    - **G** is (q x p) matrix
    - For every input, there are q transfer functions, i.e. one for each output
  - Simple representation of input-output mapping
  - Examples
    - If **B**,**C** = **I** and **A**, **D** = 0, then **y** = integral of **u**
    - If **A** = **I**, then exponential



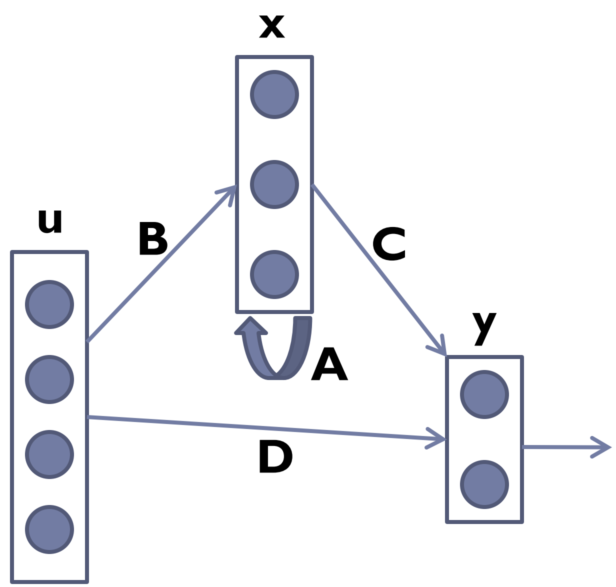
Wikipedia

State space representations

- Linear systems theory

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

  - Link to neural networks (linear time-invariant models)



State space representations

- Linear systems theory

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

  - Controllability
    - It is possible (by admissible inputs) to steer the states from any initial value to any final value within some finite time window.
    - Continuous time-invariant models are controllable if

$$rank\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix} = n$$

(rank = number of linearly independent rows in the matrix)

state variables

State space representations

- Linear systems theory

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

  - Observability
    - A measure of how well internal states of a system can be inferred by knowledge of its external outputs.
    - Continuous time-invariant models are observable if

$$rank\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \cdots \\ \mathbf{CA}^{n-1} \end{bmatrix} = n$$

Next up: Saccades